arXiv:1104.5273v2 [math-ph] 1 May 2011

Phase coherent states with circular Jacobi polynomials for the pseudoharmonic oscillator

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Abstract

We construct a class of generalized phase coherent states $\mid e^{i\theta};\gamma,\alpha,\varepsilon>$ indexed by points $e^{i\theta}$ of the unit circle and depending on three positive parameters γ, α and ε by replacing the labeling coefficient $z^n/\sqrt{n!}$ of the canonical coherent states by circular Jacobi polynomials $g_n^{\gamma}\left(e^{i\theta}\right)$ with parameter $\gamma\geq 0$. The special case of $\gamma=0$ corresponds to well known phase coherent states. The constructed states are superposition of eigenstates of a one-parameter pseudoharmonic oscillator depending on α and solve the identity of the state Hilbert space at the limit $\varepsilon\to 0^+$. Closed form for their wavefunctions are obtained in the case $\alpha=\gamma+1$ and their associated coherent states transform is defined.

1 Introduction

The first phase states which appeared in quantum mechanics were the London phase states [1]:

$$\mid e^{i\theta}>=\sum_{n=0}^{+\infty}e^{in\theta}\mid n> \tag{1.1}$$

which actually possess infinite energy and can be approached by physical states in different ways. Consequently, different phase-enhanced states have been proposed and studied in the literature; see [2, 3] and references therein.

The phase analogue of coherent states have been introduced as eigenstates of the Susskind-Glogower operator [4]:

$$\mathcal{E}^{SG}:=\sum_{n=0}^{+\infty}\mid n><n+\mid . \tag{1.2}$$

There are the so-called phase coherent states (PCS), whose expression in the Fock basis is given by

$$\mid e^{i\theta}>=\left(1-{}^2\right)^{\frac{1}{2}}\sum_{n=0}^{+\infty}{}^n e^{in\theta}\mid n>;\quad 0\leq\rho<1 \tag{1.3}$$

and represent realistic states of radiation, namely they possess finite energy and can be synthesized by suitable nonlinear process [5]. They can also be viewed as a special case of negative binomial states [6] or Perelomov's $su(1,1)$ coherent states via its Holstein-Primakoff realization with $\frac{1}{2}$ as Bargmann index [7].

In this paper, we construct a class of generalized phase coherent states (GPCS) labeled by points $e^{i\theta}$ of the unit circle S^1 and depending on three positive parameters: γ, α and ε. These states belong to the state Hilbert space $L^2(\mathbb{R}_+, dx)$ of the Hamiltonian with pseudoharmonic oscillator potential (PHO) given by ([8]):

$$\Delta_a := -\frac{d^2}{dx^2} + x^2 + \frac{a}{x^2}, \tag{1.4}$$

where $a > 0$ is such that $1 + \frac{1}{2}\sqrt{1+4a} = \alpha$.

We precisely adopt a formalism of canonical coherent states when written as superpositions of the harmonic oscillator number states. That is, we present a GPCS as a superposition of eigenstates of the Hamiltonian in (1.4). In this superposition, the role of coefficients $z^n/\sqrt{n!}$ is played by the circular Jacobi polynomials ([9, p.230], [10, 11]) given, up to a normalization, by:

$$g_n^{\gamma}\left(e^{i\theta}\right) := (n!)^{-1}(\gamma+1)_n\, {}_2F_1(-n, \frac{\gamma}{2}+1, \gamma+1, 1-e^{i\theta}) \tag{1.5}$$

where ${}_2F_1$ is the Gauss hypergeometric function and $(\cdot)_n$ denotes Pochhammer symbol. These orthogonal polynomials arise in a class of random matrix ensembles, where the parameter γ is related to the charge of an impurity fixed at $z = 1$ in a system of unit charges located on S^1 at the complex values given by eigenvalues of a member of this matrix ensemble [9], [12].

The class of GPCS we are introducing contains the form of the well known PCS in (1.3) as the special case $\gamma = 0$. The identity of the state Hilbert space $L^2(\mathbb{R}_+, dx)$ carrying the GCPS is solved at the limit $\varepsilon \to 0^+$ by a similar way in a previous work [13]. Furthermore, if we link γ with the parameter α controlling the singular part of PHO potential in (1.4) by $\alpha = \gamma + 1$, then we can establish a closed form for the constructed states. In this case, we propose a suitable definition for the associated coherent states transform and we check that it maps the eigenstates of the Hamiltonian Δ_a, which are defined $\mathbb{R}_+$ onto the normalized circular Jacobi polynomials defined on S^1.

The paper is organized as follows. In Section 2, we recall briefly some needed spectral properties of the Hamiltonian with PHO potential. Section 3 is devoted to the coherent states formalism we will be using. This formalism is applied in Section 4 so as to construct a class of phase coherent states in the state Hilbert space of the Hamiltonian. In Section 5 we give a closed form for these states an we discuss their associated coherent states transform.

2 The pseudoharmonic oscillator

The PHO potential was pointed out in [8] and [14] and studied by many authors (see [15] and references therein). It can be used to calculate the vibrational energies of a diatomic molecules, with the form

$$V_{\varrho,\kappa_0}(x) := \varrho\left(\frac{x}{\kappa_0} - \frac{\kappa_0}{x}\right)^2, \tag{2.1}$$

where $\kappa_0 > 0$ denotes the equilibrium bond length which is the distance between the diatomic nuclei, and $\varrho > 0$ with $F = \varrho\kappa_0^{-2}$ represents a constant force. The associated stationary Schrödinger equation reads

$$-\frac{d^2}{dx^2}\psi(x) + \varrho\left(\frac{x}{\kappa_0} - \frac{\kappa_0}{x}\right)^2\psi(x) = \lambda\psi(x), \tag{2.2}$$

with $\psi(0) = 0$, namely ψ satisfies the Dirichlet boundary condition. It is an exactly solvable equation. Indeed, according to [16, p.11288]) the energy spectrum is given by

$$\lambda_n^{\varrho,\kappa_0} := 4\kappa_0^{-1}\sqrt{\varrho}\left(n + \frac{1}{2} + \frac{1}{4}\left(\sqrt{1+4\varrho\kappa_0^2} - 2\kappa_0\sqrt{\varrho}\right)\right), n = 0, 1, 2, \cdots \tag{2.3}$$

whereas the wave functions of the exact solutions of (2.2) take the form

$$< x \mid n; \varrho, \kappa_0 > \propto x^q \exp\left(-\frac{\sqrt{\varrho}}{2\kappa_0}x^2\right) {}_1F_1\left(-n, q+1, \frac{\sqrt{\varrho}}{2\kappa_0}x^2\right), \tag{2.4}$$

where $q = \frac{1}{2}\left(1 + \sqrt{1+4\varrho\kappa_0^2}\right)$ and ${}_1F_1$ denotes the confluent hypergeometric function which can also be expressed in terms of Laguerre polynomials as ([17, p.240]):

$${}_1F_1(-n, \nu, u) = \frac{n!}{(\nu)_n} L_n^{(\nu-1)}(u), \tag{2.5}$$

where the Pochhammer symbol may also be defined by the Euler gamma function as

$$(\nu)_0 = 1, (\nu)_n = \nu(\nu+1)\cdots(\nu+n-1) = \frac{\Gamma(\nu+n)}{\Gamma(\nu)}; \quad n = 1, 2, \cdots. \tag{2.6}$$

To simplify the notation, we introduce the parameters $a := \varrho\kappa_0^2$ and we put $\kappa_0^{-1}\sqrt{\varrho} = 1$, and thereby the Hamiltonian in (2.2) takes the form given in (1.4) by the operator Δ_a which is also called isotonic oscillator [18] or Gol'dman-Krivchenkov Hamiltonian [16]. The spectrum of Δ_a in the Hilbert space $L^2(\mathbb{R}_+, dx)$ reduces to its discrete part consisting of eigenvalues of the form ([19, pp.9-10]):

$$\lambda_n^\alpha := 2(2n+\alpha), \alpha = 1 + \frac{1}{2}\sqrt{1+4a}; \quad n = 0, 1, 2, \cdots, \tag{2.7}$$

and wavefunctions of the corresponding normalized eigenfunctions are given by

$$< x \mid n; \alpha > := \left(\frac{2n!}{\Gamma(\alpha+n)}\right)^{\frac{1}{2}} x^{\alpha-\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(\alpha-1)}\left(x^2\right); \quad n = 0, 1, 2, \cdots. \tag{2.8}$$

The set of functions in (2.8) constitutes a complete orthonormal basis for the Hilbert space $L^2(\mathbb{R}_+, dx)$.

Remark 2.1 *We should note that the eigenvalue problem for the PHO can also be considered by using raising and lowering operators throughout a factorization of the Hamiltonian Δ_a in* (1.4) *based on the Lie algebra $su(1,1)$ commutation relations [15].*

3 A coherent states formalism

In general, coherent states are a specific overcomplete family of vectors in the Hilbert space of the problem that describes the quantum phenomena and solves the identity of this Hilbert space. These states have long been known for the harmonic oscillator and their properties have frequently been taken as models for defining this notion for other models [20]. In this section, we adopt the generalization of canonical coherent states as in [13], which extend a well known generalization [18] by considering a kind of the identity resolution that we obtain as a limit with respect to a certain parameter. Precisely, we propose the following formalism.

Definition 3.1 *Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{\psi_n\}_{n=0}^{+\infty}$. Let $\mathfrak{D} \subseteq \mathbb{C}$ be an open subset of $\mathbb{C}$ and let $\Phi_n : \mathfrak{D} \to \mathbb{C}; \quad n = 0, 1, 2, \cdots$, be a sequence of complex functions. Define*

$$| z, \varepsilon >:= (N_\varepsilon(z))^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{\Phi_n(z)}{\sqrt{\sigma_\varepsilon(n)}} | \psi_n >; \quad z \in \mathfrak{D}, \varepsilon > 0, \tag{3.1}$$

where $N_\varepsilon(z)$ is a normalization factor and $\sigma_\varepsilon(n)$; $n = 0, 1, 2, \cdots$, a sequence of positive numbers depending on $\varepsilon > 0$. The set of vectors $\{| z, \varepsilon >, z \in \mathfrak{D}\}$ is said to form a set of generalized coherent states if :
(i) for each fixed $\varepsilon > 0$ and $z \in \mathfrak{D}$, the state $| z, \varepsilon >$ is normalized, that is $< z, \varepsilon | z, \varepsilon >_{\mathcal{H}} = 1$,
(ii) the states $\{| z, \varepsilon >, z \in \mathfrak{D}\}$ satisfy the following resolution of the identity

$$\lim_{\varepsilon \to 0^+} \int_{\mathfrak{D}} | z, \varepsilon >< z, \varepsilon | \, d\mu_\varepsilon(z) = \mathbf{1}_{\mathcal{H}} \tag{3.2}$$

where $d\mu_\varepsilon$ is an appropriately chosen measure and $\mathbf{1}_{\mathcal{H}}$ is the identity operator on the Hilbert space $\mathcal{H}$.

We should precise that, in the above definition, the Dirac's *bra-ket* notation $| z, \varepsilon >< z, \varepsilon |$ means the rank-one-operator $\varphi \longmapsto < \varphi | z, \varepsilon >_{\mathcal{H}} | z, \varepsilon >$, $\varphi \in \mathcal{H}$. Also, the limit in (ii) is to be understood as follows. Define the operator

$$\mathcal{O}_\varepsilon[\varphi](\cdot) := \left(\int_{\mathfrak{D}} | z, \varepsilon >< z, \varepsilon | \, d\mu_\varepsilon(z) \right) [\varphi](\cdot) \tag{3.3}$$

then the above limit (3.2) means that $\mathcal{O}_\varepsilon[\varphi](\cdot) \to \varphi(\cdot)$ as $\varepsilon \to 0^+$, *almost every where* with respect to $(\cdot)$.

Remark 3.2 *The formula* (3.1) *can be considered as a generalization of the series expansion of the canonical coherent states*

$$| z >:= \left(e^{|z|^2} \right)^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \frac{z^k}{\sqrt{k!}} | k >; \quad z \in \mathbb{C}. \tag{3.4}$$

with $| k >$; $k = 0, 1, 2, \cdots$, being an orthonormal basis in $L^2(\mathbb{R}, d\xi)$ of eigenstates of the harmonic oscillator, which is given by the wavefunctions are $< \xi | k >:= \left(\sqrt{\pi} 2^k k! \right)^{-\frac{1}{2}} e^{-\frac{1}{2}\xi^2} H_k(\xi)$ where $H_k(.)$ denotes the kth Hermite polynomial [17].

4 Generalized phase coherent states

We now will construct a set of normalized states labeled by points $e^{i\theta}$ of the unit circle $S^1 = \{\omega \in \mathbb{C}, |\omega| = 1\}$ and depending on positive parameters γ, α and $\varepsilon > 0$. These states, denoted by $| e^{i\theta}; \varepsilon, \gamma, \alpha >$, will belong to $L^2(\mathbb{R}_+, dx)$ the state Hilbert space of the Hamiltonian Δ_a in (1.4) as mentioned in the introduction.

Definition 4.1 *Define a set of states $| e^{i\theta}; \varepsilon, \gamma, \alpha >$ labeled by points $e^{i\theta} \in S^1, \theta \in [0, 2\pi]$ and depending on the parameters $\gamma \geq 0$, $\alpha > \frac{3}{2}$ and $\varepsilon > 0$ by*

$$| e^{i\theta}; \varepsilon, \gamma, \alpha >:= (\mathcal{N}_{\gamma,\varepsilon}(\theta))^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{g_n^\gamma(e^{i\theta})}{\sqrt{\sigma_{\gamma,\varepsilon}(n)}} | n; \alpha > \tag{4.1}$$

with the precisions:

- *$\mathcal{N}_{\gamma,\varepsilon}(\theta)$ is a normalization factor such that $< e^{i\theta},\varepsilon,\gamma,\alpha \mid e^{i\theta},\varepsilon,\gamma,\alpha >= 1$*
- *$g_m^{\gamma}\left(e^{i\theta}\right)$ are the circular Jacobi polynomials defined in* (1.5)
- *$\sigma_{\gamma,\varepsilon}(n)$, $n = 0,1,2,\cdots$ are a sequence of positive numbers given by*

$$\sigma_{\gamma,\varepsilon}(n) := (n!)^{-1}(\gamma+1)_n e^{n\varepsilon}, \tag{4.2}$$

- *$\mid n;\alpha >; \quad n = 0,1,2,\cdots$, is the orthonormal basis of $L^2(\mathbb{R}_+, dx)$ given in* (2.6).

We shall give the main properties on these states in two propositions.

Proposition 4.2 *Let $\gamma \geq 0$ and $\varepsilon > 0$ be fixed parameters. Then, the normalization factor in* (4.1) *has the expression*

$$\mathcal{N}_{\gamma,\varepsilon}(\theta) = \frac{(1-e^{-\varepsilon})}{\left|1-e^{-\varepsilon+i\theta}\right|^{2+\gamma}}\, {}_2F_1\left(\frac{\gamma}{2}+1, \frac{\gamma}{2}+1, \gamma+1; \frac{e^{-\varepsilon}\left|1-e^{i\theta}\right|^2}{\left|1-e^{-\varepsilon+i\theta}\right|^2}\right) \tag{4.3}$$

for every $\theta \in [0,2\pi]$.

Proof. To calculate this factor, we start by writing the condition

$$1 =< e^{i\theta},\varepsilon,\gamma,\alpha \mid e^{i\theta},\varepsilon,\gamma,\alpha > . \tag{4.4}$$

Eq.(4.4) is equivalent to

$$\left(\mathcal{N}_{\gamma,\varepsilon}(\theta)\right)^{-1} \sum_{n=0}^{+\infty} \frac{1}{\sigma_{\gamma,\varepsilon}(n)} g_n^{\gamma}\left(e^{i\theta}\right) \overline{g_n^{\gamma}\left(e^{i\theta}\right)} = 1. \tag{4.5}$$

Inserting the expression (4.2) into (4.5), we obtain that

$$\mathcal{N}_{\gamma,\varepsilon}(\theta) = \sum_{n=0}^{+\infty} \frac{n! e^{-n\varepsilon}}{(\gamma+1)_n} g_n^{\gamma}\left(e^{i\theta}\right) \overline{g_n^{\gamma}\left(e^{i\theta}\right)}. \tag{4.6}$$

Explicitly, the sum in (4.6) reads

$$\sum_{n=0}^{+\infty} \frac{(\gamma+1)_n}{n! e^{n\varepsilon}}\, {}_2F_1\left(-n, \frac{\gamma}{2}+1, \gamma+1, 1-e^{i\theta}\right)\, {}_2F_1\left(-n, \frac{\gamma}{2}+1, \gamma+1, 1-e^{-i\theta}\right). \tag{4.7}$$

Making use of the formula ([21, p.85]):

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{(c)_n r^n}{n!}\, {}_2F_1(-n,a,c,\xi)\, {}_2F_1(-n,b,c,\zeta) &= \frac{(1-r)^{a+b-c}}{(1-r+\xi r)^a (1-r+\zeta r)^b} \\ &\quad \times {}_2F_1\left(a,b,c; \frac{r\xi\zeta}{(1-r+\xi r)(1-r+\zeta r)}\right) \end{aligned} \tag{4.8}$$

for $c = \gamma+1$, $r = e^{-\varepsilon}$, $a = b = \frac{\gamma}{2}+1$, $\xi = 1-e^{i\theta}$ and $\zeta = \overline{\xi}$, we arrive at the expression

$$\mathcal{N}_{\gamma,\varepsilon}(\theta) = \frac{(1-e^{-\varepsilon})}{\left|1-e^{-\varepsilon+i\theta}\right|^{2+\gamma}}\, {}_2F_1\left(\frac{\gamma}{2}+1, \frac{\gamma}{2}+1, \gamma+1; \frac{e^{-\varepsilon}\left|1-e^{i\theta}\right|^2}{\left|1-e^{-\varepsilon+i\theta}\right|^2}\right) \tag{4.9}$$

This ends the proof. ■

As mentioned in the introduction, if we consider the case $\gamma = 0$, then one can check that the quantity in (4.9) equals to $\mathcal{N}_{0,\varepsilon}(\theta) = (1-e^{-\varepsilon})^{-1}$and the sequence of numbers in (4.2) become $\sigma_{0,\varepsilon}(n)$ $=e^{n\varepsilon}$ while the circular Jacobi polynomials in (1.5) reduces to

$$g_n^0\left(e^{i\theta}\right) = \frac{(1)_n}{n!}{}_2F_1(-n,1,1,1-e^{i\theta}) = e^{in\theta}. \tag{4.10}$$

Therefore, setting $\rho = e^{-\frac{1}{2}\varepsilon} < 1$, the constructed states take the form:

$$\mid e^{i\theta},\varepsilon(\rho),0,\alpha >= \sqrt{1-\rho^2}\sum_{n=0}^{+\infty}\rho^n e^{in\theta} \mid n,\alpha > . \tag{4.11}$$

The latter can be considered as a phase coherent state expressed in the basis of eigenstates of the Hamiltonian with PHO potential, whose form is identical to the PCS in (1.3).

Proposition 4.3 *The states* $\mid e^{i\theta},\varepsilon,\gamma,\alpha >$ *satisfy the following resolution of the identity*

$$\lim_{\varepsilon\to 0^+}\int_0^{2\pi} \mid e^{i\theta},\varepsilon,\gamma,\alpha >< \alpha,\gamma,\varepsilon,e^{i\theta} \mid d\mu_{\gamma,\varepsilon}(\theta) = \mathbf{1}_{L^2(\mathbb{R}_+,dx)} \tag{4.12}$$

where $\mathbf{1}_{L^2(\mathbb{R}_+,dx)}$ *is the identity operator and* $d\mu_{\gamma,\varepsilon}(\theta)$ *is a measure on* $[0,2\pi]$ *with the expression*

$$d\mu_{\gamma,\varepsilon}(\theta) := 2^\gamma\frac{\Gamma^2\left(\frac{1}{2}\gamma+1\right)}{\Gamma(\gamma+1)}\left(\sin\frac{\theta}{2}\right)^\gamma \mathcal{N}_{\gamma,\varepsilon}(\theta)\frac{d\theta}{2\pi}, \tag{4.13}$$

$\mathcal{N}_{\gamma,\varepsilon}(\theta)$ *being the normalization factor given explicitly in* (4.9).

Proof. Let us assume that the measure takes the form

$$d\mu_{\gamma,\varepsilon}(\theta) = \mathcal{N}_{\gamma,\varepsilon}(\theta)\,\Omega_\gamma(\theta)\,d\theta \tag{4.14}$$

where $\Omega_\gamma(\theta)$ is an auxiliary density to be determined. Let $\varphi \in L^2(\mathbb{R}_+,dx)$ and let us start by writing the following action

$$\mathcal{O}_{\gamma,\varepsilon}[\varphi] := \left(\int_0^{2\pi} \mid e^{i\theta},\varepsilon,\gamma,\alpha >< \alpha,\gamma,\varepsilon,e^{i\theta} \mid d\mu_{\gamma,\varepsilon}(\theta)\right)[\varphi] \tag{4.15}$$

$$= \int_0^{2\pi} < \varphi \mid e^{i\theta};\varepsilon,\gamma,\alpha >< \alpha,\gamma,\varepsilon,e^{i\theta} \mid d\mu_{\gamma,\varepsilon}(\theta). \tag{4.16}$$

Making use Eq. (4.1), we obtain successively

$$\mathcal{O}_{\gamma,\varepsilon}[\varphi] = \int_0^{2\pi} < \varphi \mid (\mathcal{N}_{\gamma,\varepsilon}(\theta))^{-\frac{1}{2}}\sum_{n=0}^{+\infty}\frac{\overline{g_n^\gamma(e^{i\theta})}}{\sqrt{\sigma_{\gamma,\varepsilon}(n)}} \mid n;\alpha >>< \alpha,\gamma,\varepsilon,e^{i\theta} \mid d\mu_{\gamma,\varepsilon}(\theta) \tag{4.17}$$

$$= \int_0^{2\pi}\sum_{n=0}^{+\infty}\frac{\overline{g_n^\gamma(e^{i\theta})}}{\sqrt{\sigma_{\gamma,\varepsilon}(n)}} < \varphi \mid n;\alpha >< \alpha,\gamma,\varepsilon,e^{i\theta} \mid (\mathcal{N}_{\gamma,\varepsilon}(\theta))^{-\frac{1}{2}}\,d\mu_{\gamma,\varepsilon}(\theta) \tag{4.18}$$

$$= \left(\sum_{m,n=0}^{+\infty}\int_0^{2\pi}\frac{\overline{g_n^\gamma(e^{i\theta})}g_m^\gamma\left(e^{i\theta}\right)}{\sqrt{\sigma_{\gamma,\varepsilon}(n)}\sqrt{\sigma_{\gamma,\varepsilon}(m)}} \mid n;\alpha >< \alpha;m \mid (\mathcal{N}_{\gamma,\varepsilon}(\theta))^{-1}\,d\mu_{\gamma,\varepsilon}(\theta)\right)[\varphi]. \tag{4.19}$$

Replace $d\mu_{\gamma,\varepsilon}(\theta)=\mathcal{N}_{\gamma,\varepsilon}(\theta)\,\Omega_{\gamma}(\theta)\,d\theta$, then Eq. (4.19) takes the form

$$\mathcal{O}_{\gamma,\alpha,\varepsilon}=\sum_{m,n=0}^{+\infty}\left[\int_{0}^{2\pi}\frac{\overline{g_{n}^{\gamma}\left(e^{i\theta}\right)}g_{m}^{\gamma}\left(e^{i\theta}\right)}{\sqrt{\sigma_{\gamma,\varepsilon}(n)}\sqrt{\sigma_{\gamma,\varepsilon}(m)}}\Omega_{\gamma}(\theta)\,d\theta\right]\mid n;\alpha><\alpha;m\mid. \tag{4.20}$$

Then, we need to consider the integral

$$I_{n,m}(\gamma,\varepsilon):=\frac{1}{\sqrt{\sigma_{\gamma,\varepsilon}(n)}\sqrt{\sigma_{\gamma,\varepsilon}(m)}}\int_{0}^{2\pi}\overline{g_{n}^{\gamma}\left(e^{i\theta}\right)}g_{m}^{\gamma}\left(e^{i\theta}\right)\Omega_{\gamma}(\theta)\,d\theta. \tag{4.21}$$

We recall the orthogonality relations of circular Jacobi polynomials [10, p.875]):

$$2^{\gamma}\frac{\Gamma^{2}\left(\frac{\gamma}{2}+1\right)}{2\pi}\int_{0}^{2\pi}\overline{g_{n}^{\gamma}\left(e^{i\theta}\right)}g_{m}^{\gamma}\left(e^{i\theta}\right)\left(\sin\frac{\theta}{2}\right)^{\gamma}dx=\frac{\Gamma(n+\gamma+1)}{n!}\delta_{n,m}. \tag{4.22}$$

This suggests us to set

$$\Omega_{\gamma}(\theta):=2^{\gamma}\frac{\Gamma^{2}\left(\frac{\gamma}{2}+1\right)}{\Gamma(\gamma+1)}\left(\sin\frac{\theta}{2}\right)^{\gamma}\frac{d\theta}{2\pi}. \tag{4.23}$$

Therefore, (4.21) reduces to

$$I_{n,m}(\gamma,\varepsilon)=e^{-\varepsilon n}\frac{m!\Gamma(n+\gamma+1)}{n!\Gamma(m+\gamma+1)}\delta_{n,m} \tag{4.24}$$

which means that the operator in (4.15) takes the form:

$$\mathcal{O}_{\gamma,\alpha,\varepsilon}\equiv\mathcal{O}_{\alpha,\varepsilon}=\sum_{n,m=0}^{+\infty}e^{-n\varepsilon}\frac{m!\Gamma(n+\gamma+1)}{n!\Gamma(m+\gamma+1)}\delta_{n,m}\mid n;\alpha><\alpha;m\mid \tag{4.25}$$

$$=\sum_{m=0}^{+\infty}e^{-m\varepsilon}\mid m;\alpha><\alpha;m\mid. \tag{4.26}$$

Thus, we arrive at

$$\mathcal{O}_{\alpha,\varepsilon}[\varphi]=\sum_{m=0}^{+\infty}e^{-m\varepsilon}\left(\mid m;\alpha><\alpha;m\mid\right)[\varphi]. \tag{4.27}$$

For $u\in\mathbb{R}_{+}$, we can write

$$\mathcal{O}_{\alpha,\varepsilon}[\varphi](u)=\sum_{m=0}^{+\infty}e^{-m\varepsilon}<\varphi\mid m;\alpha><u\mid m;\alpha> \tag{4.28}$$

$$=\sum_{m=0}^{+\infty}e^{-m\varepsilon}\left(\int_{0}^{+\infty}\varphi(v)\overline{<v\mid m;\alpha>}dv\right)<u\mid m;\alpha> \tag{4.29}$$

$$=\int_{0}^{+\infty}\varphi(v)\left(\sum_{m=0}^{+\infty}e^{-m\varepsilon}\overline{<v\mid m;\alpha>}<u\mid m;\alpha>\right)dv. \tag{4.30}$$

We are then lead to calculate the sum

$$\mathcal{G}_{\varepsilon}^{\alpha}(u,v) := \sum_{m=0}^{+\infty} e^{-m\varepsilon}\overline{< v \mid m;\alpha >} < u \mid m;\alpha > . \tag{4.31}$$

For this we recall the expression of the eigenstate $\mid m;\alpha >$ in (2.8). So that the above sum in (4.31) reads

$$\mathcal{G}_{\varepsilon}^{\alpha}(u,v) = 2\,(vu)^{\alpha-\frac{1}{2}}\, e^{-\frac{1}{2}\left(u^2+v^2\right)} \sum_{m=0}^{+\infty} e^{-m\varepsilon}\frac{m!}{\Gamma(m+\alpha)} L_m^{(\alpha-1)}\left(u^2\right) L_m^{(\alpha-1)}\left(v^2\right). \tag{4.32}$$

Eq.(4.32) can be rewritten as

$$\mathcal{G}_{\varepsilon}^{\alpha}(u,v) = 2\,(uv)^{\alpha-\frac{1}{2}}\, e^{-\frac{1}{2}\left(u^2+v^2\right)} K\left(e^{-\varepsilon}; u^2, v^2\right) \tag{4.33}$$

where we have introduced the kernel function

$$K(\tau;\xi,\zeta) := \sum_{m=0}^{+\infty} \tau^m \frac{m!}{\Gamma(m+\alpha)} L_m^{(\alpha-1)}(\xi)\, L_m^{(\alpha-1)}(\zeta)\,; \quad 0 < \tau < 1. \tag{4.34}$$

The latter can be written in a closed form by applying the Hille-Hardy formula [17]. Now, returning back to Eq. (4.30) and taking into account Eq.(4.33), we get that

$$\mathcal{O}_{\alpha,\varepsilon}[\varphi](u) = 2u^{\alpha-\frac{1}{2}} e^{-\frac{1}{2}u^2} \int_0^{+\infty} v^{\alpha-\frac{1}{2}} e^{-\frac{1}{2}v^2} K\left(e^{-\varepsilon}, u^2, v^2\right) \varphi(v)\, dv. \tag{4.35}$$

Next,we split the right hand side of Eq.(4.35)

$$\mathcal{O}_{\alpha,\varepsilon}[\varphi](u) = \vartheta_\alpha(u)\, M[\varphi](u), \tag{4.36}$$

where

$$M[\varphi](u) = \frac{1}{2}\int_0^{+\infty} K(\tau;w,s)\, h(s)\, s^{\alpha-1} e^{-s} ds, \tag{4.37}$$

with $\tau = e^{-\varepsilon}$, $w = u^2$ and

$$h(s) := s^{-\frac{1}{2}\alpha+\frac{1}{4}} e^{\frac{1}{2}s} \varphi\left(\sqrt{s}\right). \tag{4.38}$$

By direct calculations, one can check that $h \in L^2\left(\mathbb{R}_+, s^{\alpha-1}e^{-s}ds\right)$. Precisely, we have that

$$\|h\|^2_{L^2(\mathbb{R}_+, s^{\alpha-1}e^{-s}ds)} = 2\,\|\varphi\|_{L^2(\mathbb{R}_+)} \tag{4.39}$$

We now apply the result of B. Muckenhoopt [22] who considered the Poisson integral of a function $f \in L^p\left(\mathbb{R}^+, s^\eta e^{-s} ds\right), \eta > -1, 1 \le p \le +\infty$ defined by

$$A[f](\tau,w) := \int_0^{+\infty} K(\tau,w,s)\, f(s)\, s^\eta e^{-s} ds; \quad 0 < \tau < 1 \tag{4.40}$$

with the kernel $K(\tau,\bullet,\bullet)$ defined as in (4.34). He proved that $\lim_{\tau\to 1^-} A[f](\tau,y) = f(y)$ almost everywhere in $[0,+\infty[, 1 \le p \le \infty$. We apply this result in the case $p = 2, f = h$ and $A \equiv M$ to obtain that

$$M[\varphi](u) \to 2^{-1} h\left(u^2\right) = 2^{-1} u^{-\alpha+\frac{1}{2}} e^{\frac{1}{2}u^2} \varphi(u). \tag{4.41}$$

Recalling that $\tau = e^{-\varepsilon}$, we get that

$$\mathcal{O}_{\alpha,\varepsilon}[\varphi](u) = \vartheta_\alpha(u) M[\varphi](u) \to \varphi(u) \quad \text{as} \quad \varepsilon \to 0^+ \tag{4.42}$$

which means that

$$\lim_{\varepsilon \to 0^+} \int_0^{2\pi} | e^{i\theta}, \varepsilon, \gamma, \alpha >< e^{i\theta}, \varepsilon, \gamma, \alpha | \, d\mu_{\gamma,\varepsilon}(\theta) = \mathbf{1}_{L^2(\mathbb{R}_+, dx)}. \tag{4.43}$$

This ends the proof. ■

5 A close form for the GPCS wavefunctions

Now, we assume that the parameter γ occurring in the definition of the circular Jacobi polynomials $g_n^\gamma(e^{i\theta})$ in (1.5) is connected to the parameter α controlling the singular part ax^{-2} of the Hamiltonian Δ_a in (1.4) by $\alpha = \gamma + 1$ which means we are taking $\gamma = \frac{1}{2}\sqrt{1+4a}$. Then, we can establish a closed form for the constructed GPCS as follows.

Proposition 5.1 *Let $\gamma = \frac{1}{2}\sqrt{1+4a}$ and $\varepsilon > 0$ be fixed parameters. Then, the wavefunctions of the states $| e^{i\theta}, \varepsilon, \gamma >$ defined in* (4.1) *can be written in a closed form as $< x | e^{i\theta}, \varepsilon, \gamma >=$*

$$\frac{\sqrt{2}\left(1 - e^{-\frac{1}{2}\varepsilon + i\theta}\right)^{-1} x^{\gamma+\frac{1}{2}} \exp\left(-\frac{x^2}{2}\coth\frac{\varepsilon}{4}\right) . {}_1F_1\left(1+\frac{\gamma}{2}, 1+\gamma; \frac{\left(1-e^{i\theta}\right)e^{-\frac{1}{2}\varepsilon}x^2}{\left(1-e^{-\frac{1}{2}\varepsilon}\right)\left(1-e^{-\frac{1}{2}\varepsilon+i\theta}\right)}\right)}{\sqrt{\Gamma(\gamma+1)}\left(\left(1-e^{-\frac{1}{2}\varepsilon}\right)\left(1-e^{-\frac{1}{2}\varepsilon+i\theta}\right)\right)^{\frac{\gamma}{2}} \sqrt{\frac{(1-e^{-\varepsilon})}{|1-e^{-\varepsilon+i\theta}|^{2+\gamma}} {}_2F_1\left(\frac{\gamma}{2}+1, \frac{\gamma}{2}+1, \gamma+1; \frac{e^{-\varepsilon}|1-e^{i\theta}|^2}{|1-e^{-\varepsilon+i\theta}|^2}\right)}} \tag{5.1}$$

for every $x \in \mathbb{R}_+$.

Proof. We start by writing the expression of the wave function of states $| e^{i\theta}, \varepsilon, \gamma >$ according to Definition (4.1) as

$$< x | e^{i\theta}, \varepsilon, \gamma >= \left(\mathcal{N}_{\gamma,\varepsilon}(\theta)\right)^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{g_n^\gamma(e^{i\theta})}{\sqrt{\sigma_{\gamma,\varepsilon}(n)}} < x | n; \gamma+1 >; \quad x \in \mathbb{R}_+. \tag{5.2}$$

We have thus to look for a closed form of the series

$$\mathcal{S}(x) := \sum_{n=0}^{+\infty} \frac{g_n^\gamma(e^{i\theta})}{\sqrt{\sigma_{\gamma,\varepsilon}(n)}} < x | n; \gamma+1 > \tag{5.3}$$

which also reads

$$\mathcal{S}(x) = \sum_{n=0}^{+\infty} \frac{(\gamma+1)_n}{n!\sqrt{\sigma_{\gamma,\varepsilon}(n)}} {}_2F_1(-n, \frac{\gamma}{2}+1, \gamma+1, 1-e^{i\theta}) < x | n; \gamma+1 > . \tag{5.4}$$

Replacing $\sigma_{\gamma,\varepsilon}(n)$ and $< x | n; \alpha >$ by their expressions in (4.1) and (2.8) respectively

$$\mathcal{S}(x) = \frac{\sqrt{2}x^{\gamma+\frac{1}{2}}e^{-\frac{1}{2}x}}{\sqrt{\Gamma(\gamma+1)}} \sum_{n=0}^{+\infty} e^{-\frac{1}{2}n\varepsilon} . {}_2F_1(-n, \frac{\gamma}{2}+1, \gamma+1, 1-e^{i\theta}) L_n^{(\gamma)}(x^2). \tag{5.5}$$

Put $\tau := e^{-\frac{1}{2}\varepsilon}, |\tau| < 1$. Then Equation (5.5) becomes

$$\mathcal{S}(x) = \frac{\sqrt{2}x^{\gamma+\frac{1}{2}}e^{-\frac{1}{2}x^2}}{\sqrt{\Gamma(\gamma+1)}}\mathfrak{S}(x), \tag{5.6}$$

where

$$\mathfrak{S}(x) := \sum_{n=0}^{+\infty} \tau^n \, {}_2F_1(-n, \frac{\gamma}{2}+1, \gamma+1, 1-e^{i\theta})L_n^{(\gamma)}\left(x^2\right). \tag{5.7}$$

Next, we use the bilateral generating formula [23, p.213]:

$$\sum_{n=0}^{+\infty} t^n \, {}_2F_1(-n, c, 1+\nu; y)L_n^{(\nu)}(u) = (1-t)^{-1+c-\nu}(1-t+yt)^{-c} \tag{5.8}$$

$$\times \exp\left(\frac{-ut}{1-t}\right) {}_1F_1\left(c, 1+\nu, \frac{yut}{(1-t)(1-t+yt)}\right)$$

for $t = \tau, c = \frac{\gamma}{2}+1, y = 1-e^{i\theta}, \nu = \gamma$ and $u = x^2$, we obtain

$$\begin{aligned} \mathfrak{S}(x) &= (1-\tau)^{-\frac{1}{2}\gamma}\left(1-\tau e^{i\theta}\right)^{-1-\frac{1}{2}\gamma} \\ &\times \exp\left(\frac{-\tau x^2}{1-\tau}\right) \cdot {}_1F_1\left(1+\frac{\gamma}{2}, 1+\gamma, \frac{\left(1-e^{i\theta}\right)\tau x^2}{(1-\tau)\left(1-\tau e^{i\theta}\right)}\right). \end{aligned} \tag{5.9}$$

Summarizing up the above calculations as

$$< x \mid e^{i\theta}, \varepsilon, \gamma >= \left(\mathcal{N}_{\gamma,\varepsilon}(\theta)\right)^{-\frac{1}{2}} \frac{\sqrt{2}x^{\gamma+\frac{1}{2}}e^{-\frac{1}{2}x^2}}{\sqrt{\Gamma(\gamma+1)}}\mathfrak{S}(x), \tag{5.10}$$

we arrive at

$$\begin{aligned} < x \mid e^{i\theta}, \varepsilon, \gamma > &= \frac{\sqrt{2}x^{\gamma+\frac{1}{2}}e^{-\frac{1}{2}x^2}(1-\tau)^{-\frac{1}{2}\gamma}}{\sqrt{\Gamma(\gamma+1)}\left(\mathcal{N}_{\gamma,\varepsilon}(\theta)\right)^{\frac{1}{2}}\left(1-\tau e^{i\theta}\right)^{1+\frac{1}{2}\gamma}} \\ &\times \exp\left(\frac{-\tau x^2}{1-\tau}\right) \cdot {}_1F_1\left(1+\frac{\gamma}{2}, 1+\gamma, \frac{\left(1-e^{i\theta}\right)\tau x^2}{(1-\tau)\left(1-\tau e^{i\theta}\right)}\right) \end{aligned} \tag{5.11}$$

Finally, we replace τ by $e^{-\frac{1}{2}\varepsilon}$ and the factor $\mathcal{N}_{\gamma,\varepsilon}(\theta)$ by its expression in (4.3) to obtain (5.1). ■

Naturally, once we have obtained a closed form for the GPCS $\mid e^{i\theta}, \varepsilon, \gamma >$ we can look for the associated coherent state transform. In view of (4.1), this transform should map the space $L^2(\mathbb{R}_+, dx)$ spanned by the eigenstates $\mid n; \gamma+1 >$ of the Hamiltonian in Δ_a onto the space in which the coefficients $g_n^{\gamma}\left(e^{i\theta}\right)$ are orthogonal, that is the space $L^2\left(S^1, d\sigma_\gamma\right)$ with $d\sigma_\gamma := \frac{1}{2\pi}\sin^\gamma \frac{\theta}{2}d\theta$. It should also obey the general form: $\varphi \mapsto \sqrt{\mathcal{N}_{\gamma,\varepsilon}(\theta)}\left\langle e^{i\theta}, \varepsilon, \gamma \mid \varphi\right\rangle$. Recalling that the resolution of the identity, which usually ensures the isometry property of such map, was obtained at the limit $\varepsilon \to 0^+$ in (4.12), then a convenient definition for such transform could be as follows.

Definition 5.2 *Let $\gamma = \frac{1}{2}\sqrt{1+4a}$ be a fixed parameter. The coherent state transform associated with the GPCS in* (4.1) *is the map*

$$\mathcal{W}_\gamma : L^2(\mathbb{R}_+, dx) \to L^2\left(S^1, d\sigma_\gamma\right) \tag{5.12}$$

defined by

$$\mathcal{W}_\gamma[\varphi]\left(e^{i\theta}\right) := \lim_{\varepsilon\to 0^+} \int_0^{+\infty} \frac{\sqrt{2}}{\sqrt{\Gamma(\gamma+1)}} x^{\gamma+\frac{1}{2}} \left(1-e^{-\frac{1}{2}\varepsilon}\right)^{-\frac{1}{2}\gamma} \left(1-e^{-\frac{1}{2}\varepsilon+i\theta}\right)^{-1-\frac{1}{2}\gamma}$$

$$\times \exp\left(-\frac{1}{2}x^2\coth\frac{\varepsilon}{4}\right) \cdot {}_1F_1\left(1+\frac{\gamma}{2}, 1+\gamma, \frac{\left(1-e^{i\theta}\right)e^{-\frac{1}{2}\varepsilon}x^2}{\left(1-e^{-\frac{1}{2}\varepsilon}\right)\left(1-e^{-\frac{1}{2}\varepsilon+i\theta}\right)}\right)\overline{\varphi(x)}dx.$$

In fact, this definition provides us with a new way of looking at the circular Jacobi polynomials. Indeed, we establish the following precise fact.

Proposition 5.3 *Let $\gamma = \frac{1}{2}\sqrt{1+4a}$ be a fixed parameter. Then, the transform $\mathcal{W}_\gamma$ defined in (5.12) satisfies:*

$$\mathcal{W}_\gamma[x \mapsto \langle x \mid n; \gamma+1\rangle]\left(e^{i\theta}\right) = \frac{\sqrt{n!}}{\sqrt{(\gamma+1)_n}} g_n^\gamma\left(e^{i\theta}\right), \tag{5.13}$$

for all $e^{i\theta} \in S^1$. In other words, the normalized circular Jacobi polynomials are the images of eigenstates of the Hamiltonian Δ_a under the coherent states transform $\mathcal{W}_\gamma$.

Proof. Let us write the transform $\mathcal{W}_\gamma$ as

$$\mathcal{W}_\gamma[x \mapsto \langle x \mid n; \gamma+1\rangle]\left(e^{i\theta}\right) = \lim_{\varepsilon\to 0^+}\sqrt{\mathcal{N}_{\gamma,\varepsilon}(\theta)}\left\langle e^{i\theta}, \varepsilon, \gamma \mid n; \gamma+1\right\rangle. \tag{5.14}$$

Next, we make use of the expression of the eigenstates in (2.8) and we shall calculate the quantity

$$\mathcal{Q}^{(\varepsilon)} := \sqrt{\mathcal{N}_{\gamma,\varepsilon}(\theta)}\left\langle e^{i\theta}, \varepsilon, \gamma \mid n; \gamma+1\right\rangle \tag{5.15}$$

in the right hand side of Eq.(5.14) without passing to the limit with respect to ε. So, for instance, we set $\tau = e^{-\frac{1}{2}\varepsilon}$ and rewrite (5.15) as

$$\mathcal{Q}^{(\varepsilon)} = \int_0^{+\infty} \frac{\sqrt{2}x^{\gamma+\frac{1}{2}}(1-\tau)^{-\frac{1}{2}\gamma}\left(1-\tau e^{i\theta}\right)^{-1-\frac{1}{2}\gamma}}{\sqrt{\Gamma(\gamma+1)}} \tag{5.16}$$

$$\times \exp\left(-\frac{1}{2}\left(\frac{1+\tau}{1-\tau}\right)x^2\right) \cdot {}_1F_1\left(1+\frac{\gamma}{2}, 1+\gamma, \frac{\left(1-e^{i\theta}\right)\tau x^2}{(1-\tau)\left(1-\tau e^{i\theta}\right)}\right)$$

$$\times \left(\frac{2n!}{\Gamma(\gamma+1+n)}\right)^{\frac{1}{2}} x^{\gamma+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(\gamma)}\left(x^2\right) dx.$$

We set

$$\kappa := \frac{\left(1-e^{i\theta}\right)\tau}{(1-\tau)\left(1-\tau e^{i\theta}\right)} \tag{5.17}$$

and we rewrite (5.16) as follows

$$\mathcal{Q}^{(\varepsilon)} = \frac{2\sqrt{n!}\,(1-\tau)^{-\frac{1}{2}\gamma}\left(1-\tau e^{i\theta}\right)^{-1-\frac{1}{2}\gamma}}{\sqrt{\Gamma(\gamma+1)}\sqrt{\Gamma(\gamma+1+n)}}\mathfrak{G}^{(\varepsilon)}, \tag{5.18}$$

where

$$\mathfrak{G}^{(\varepsilon)} \quad : \quad = \int_0^{+\infty} x^{2\gamma+1}\exp\left(-\frac{1}{1-\tau}x^2\right)L_n^{(\gamma)}\left(x^2\right)\ {}_1F_1\left(1+\frac{\gamma}{2},1+\gamma,\kappa x^2\right)dx. \tag{5.19}$$

Writing the Laguerre polynomial in term of the ${}_1F_1$ as in (2.5) and making the change of variable $t=x^2$, then (5.19) takes the form

$$\mathfrak{G}^{(\varepsilon)} = \frac{(\gamma+1)_n}{2n!}\int_0^{+\infty} t^{\gamma}e^{-\frac{1}{1-\tau}t}{}_1F_1(-n,1+\gamma,t)_1F_1\left(1+\frac{\gamma}{2},1+\gamma,\kappa t\right)dt. \tag{5.20}$$

Now, with the help of the formula ([24, p.823]):

$$\begin{aligned} & \int_0^{+\infty} t^{c-1}e^{-st}{}_1F_1(\beta,c,t)_1F_1(b,c,\lambda t)\,dt \\ = \ & \Gamma(c)(s-1)^{-\beta}(s-\lambda)^{-b}s^{\beta+b-c}\cdot{}_2F_1\left(\beta,b,c;\frac{\lambda}{(s-1)(s-\lambda)}\right), \end{aligned} \tag{5.21}$$

$\Re c>0$ and $\Re s>\Re\lambda+1$, for the parameters: $c=\gamma+1$, $\beta=-n$, $b=1+\frac{\gamma}{2}$, $s=\frac{1}{1-\tau}$ and $\lambda=\kappa$, Eq.(5.20) becomes

$$\mathfrak{G}^{(\varepsilon)} = \frac{(\gamma+1)_n\Gamma(\gamma+1)\tau^n(1-\tau)^{\gamma+1}}{2n!\,(1-\kappa(1-\tau))^{1+\frac{\gamma}{2}}}\cdot{}_2F_1\left(-n,1+\frac{\gamma}{2},1+\gamma;\frac{\kappa(1-\tau)^2}{\tau(1-\kappa(1-\tau))}\right). \tag{5.22}$$

Recalling the expression of κ in (5.17) we find that

$$\frac{\kappa(1-\tau)^2}{\tau(1-\kappa(1-\tau))} = 1-e^{i\theta}. \tag{5.23}$$

We return back to (5.18) and we obtain that

$$\begin{aligned} \mathcal{Q}^{(\varepsilon)} \quad = \quad & \tau^n\left(\frac{(\gamma+1)_n}{n!}\right)^{\frac{1}{2}}\frac{(1-\tau)^{\frac{\gamma}{2}+1}}{\left(\left(1-\tau e^{i\theta}\right)(1-\kappa(1-\tau))\right)^{1+\frac{\gamma}{2}}} \\ & \times{}_2F_1\left(-n,1+\frac{\gamma}{2},1+\gamma;1-e^{i\theta}\right) \end{aligned} \tag{5.24}$$

Using again (5.17) we find by calculation

$$\frac{(1-\tau)^{\frac{\gamma}{2}+1}}{\left(\left(1-\tau e^{i\theta}\right)(1-\kappa(1-\tau))\right)^{1+\frac{\gamma}{2}}} = 1 \tag{5.25}$$

Recalling that $\tau = e^{-\frac{1}{2}\varepsilon}$, we arrive at

$$\mathcal{Q}^{(\varepsilon)} = e^{-\frac{1}{2}n\varepsilon} \left(\frac{(\gamma+1)_n}{n!} \right)^{\frac{1}{2}} \cdot {}_2F_1 \left(-n, 1+\frac{\gamma}{2}, 1+\gamma; 1-e^{i\theta} \right). \tag{5.26}$$

If let $\varepsilon \to 0^+$, then we obtain that

$$\mathcal{W}_\gamma \left[x \mapsto \langle x \mid n; \alpha+1 \rangle \right] \left(e^{i\theta} \right) = \left(\frac{(\gamma+1)_n}{n!} \right)^{-\frac{1}{2}} g_n^\gamma \left(e^{i\theta} \right), \tag{5.27}$$

where the in the left hand side we have exactly the normalized circular Jacobi polynomials. ■

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